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A Central Extension of a Formal Loop Group

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0. Introduction

In this article, we prove that there is an elegant relation between the conformal factor and a group 2-cocycle on the formal loop group with values in $SU(1, N+1)$, and show that the trivial central extension of the Hauser group acts transitively on the space of formal solutions of the Einstein-Maxwell field equations with N abelian gauge fields. The corresponding 2-cocycle on the Lie algebra of the formal loop group is the one which describes an affine Lie algebra [K]. This relation was first found by [BM].

Now we derive the equations, which are our starting point, from the stationary axisymmetric Einstein-Maxwell field equations with N abelian gauge potentials.

Let $ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu$ be a metric on $\mathbb{R}^{1+3}$ and $\mathbf{A} = \mathbf{A}_\mu dx^\mu$ an abelian gauge potential with values in $\mathbb{R}^N$. Then the Einstein-Maxwell field equations with N abelian gauge fields are given by

$$R_{\mu\nu} = 8\pi T_{\mu\nu}, \quad \nabla_\kappa \mathbf{F}^{\mu\kappa} = 0 \quad (\mu, \nu = 0, 1, 2, 3)\ ,$$

where $R_{\mu\nu}$ is the Ricci curvature and

$$\begin{aligned} \mathbf{F}_{\mu\nu} &= \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu, \\ T_{\mu\nu} &= \frac{1}{4\pi}(\mathbf{F}_{\mu\kappa}\, {}^t\mathbf{F}_\nu{}^\kappa - \frac{1}{4} g_{\mu\nu} \mathbf{F}_{\kappa\iota}\, {}^t\mathbf{F}^{\kappa\iota}). \end{aligned}$$

We adopt the coordinates $(x^0, x^1, x^2, x^3) = (x^0, \phi, z, \rho)$ with x^0 being time and (ϕ, z, ρ) the cylindrical coordinates of $\mathbb{R}^3$. Stationary axisymmetric space-times amount to the assumption that a metric is of the form

$$g = \begin{pmatrix} h_{00} & h_{01} & & \\ h_{10} & h_{11} & & \\ & & -\lambda & 0 \\ & & 0 & -\lambda \end{pmatrix}$$

$$\det h = -\rho^2,$$

where $\lambda > 0$, $h_{01} = h_{10}$ and $h = (h_{ij})$. The field λ is called the conformal factor.

For abelian gauge potentials, we fix the gauge so as to $\mathbf{A}_2 = \mathbf{A}_3 = 0$. Since we assume that the fields are stationary and axisymmetric, the functions h_{ij}'s, λ and $\mathbf{A}_i$'s depend only on z and ρ. Further, we fix the gauge as follows :

$$h|_{(z,\rho)=(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{A}|_{(z,\rho)=(0,0)} = 0. \tag{0.1}$$

Introducing the Ernst potentials $u \in \mathbf{R}, v \in \mathbf{C}^N$ constructed from h and $\mathbf{A}$ by the standard method (cf. [DO][E]), we obtain

Proposition 0.1. *The stationary axisymmetric Einstein-Maxwell field equations with N abelian gauge fields are equivalent to the following equations :*

$$f(d * du + \rho^{-1} d\rho \wedge *du) = (du - 2v^* dv) \wedge *du \tag{0.2}$$
$$f(d * dv + \rho^{-1} d\rho \wedge *dv) = (du - 2v^* dv) \wedge *dv \tag{0.3}$$

$$\begin{aligned} \frac{\partial_z \lambda}{\lambda} = &- \frac{\partial_z f}{2f} + \frac{\rho}{2f^2}(\partial_z f \partial_\rho f) \\ &- \frac{\rho}{2f^2}(\partial_\rho u - \partial_\rho f - 2v^* \partial_\rho v)(\partial_z u - \partial_z f - 2v^* \partial_z v) \\ &+ \frac{\rho}{f}(\partial_z v^* \partial_\rho v + \partial_z v^* \partial_\rho v) \end{aligned} \tag{0.4}$$

$$\begin{aligned} \frac{\partial_\rho \lambda}{\lambda} = &- \frac{\partial_\rho f}{2f} + \frac{\rho}{4f^2}\{(\partial_\rho f)^2 - (\partial_z f)^2\} \\ &+ \frac{\rho}{4f^2}\{(\partial_z u - \partial_z f - 2v^* \partial_z v)^2 - (\partial_\rho u - \partial_\rho f - 2v^* \partial_\rho v)^2\} \\ &- \frac{\rho}{f}(\partial_z v^* \partial_z v - \partial_\rho v^* \partial_\rho v), \end{aligned} \tag{0.5}$$

where $v^* = {}^t\bar{v}, |v|^2 = v^* v, f = \mathrm{Re}\, u - |v|^2$ and $*$ is the Hodge operator given by $*dz = d\rho, *d\rho = -dz$.

The first two equations are called the Ernst equations.

Corresponding to the gauge fixing (0.1), we shall consider the solutions under the conditions

$$u|_{(z,\rho)=(0,0)} = 1 \quad \text{and} \quad v|_{(z,\rho)=(0,0)} = 0. \tag{0.6}$$

It is essential to introduce the function $\tau = f^{1/2}\lambda$ and we shall consider τ, in stead of λ, throughout this article.

1. Ernst Equation

Let θ be Cartan involution of $GL(N+2,\mathbb{C})$ defined by $g \mapsto g^{*-1}$ and G a subgroup of $GL(N+2,\mathbb{C})$ defined by

$$\{g \in GL(N+2,\mathbb{C})\,;g^*Jg = J,\ \det g = 1\},$$

where $J = \begin{pmatrix} & & i \\ & 1_N & \\ -i & & \end{pmatrix}$ and 1_N denotes the $N \times N$ identity matrix. Note that G is isomorphic to $SU(1, N+1)$. Let K be the subgroup of G such that each element of K is fixed by θ.

We fix subgroups A and N of G as follows :

$$A = \left\{ \begin{pmatrix} a & & \\ & 1_N & \\ & & 1/a \end{pmatrix} ; a > 0 \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & & \\ v & 1_N & \\ x + i|v|^2/2 & iv^* & 1 \end{pmatrix} ; x \in \mathbb{R}, v \in \mathbb{C}^N \right\},$$

where $|v|^2 = v^*v$. Then we have $G = KAN$ (Iwasawa decomposition).

Let R be a ring of formal power series in z and ρ over $\mathbb{C}$ i.e. $R = \mathbb{C}[[z,\rho]]$. We extend the complex conjugation * of $\mathbb{C}$ to a conjugation of R by defining $\bar{z} = z, \bar{\rho} = \rho$. Let G_R be a subgroup of $GL(N+2, R)$ defined by

$$\{g \in GL(N+2, R)\,;g^*Jg = J,\ \det g = 1\}.$$

Then, corresponding to $G = KAN$, G_R decomposes as $G_R = K_R A_R N_R$, where K_R, A_R and N_R denote subgroups of G_R consisting of matrices with values in K, A and N respectively, each of whose components is an element of R.

Now we parametrize an element of $A_R N_R$ as follows :

$$P = \begin{pmatrix} f^{1/2} & 0 & 0 \\ \sqrt{2}v & 1_N & 0 \\ (\psi + i\,|v|^2)/f^{1/2} & \sqrt{2}iv^*/f^{1/2} & f^{-1/2} \end{pmatrix}, \tag{1.1}$$

where f and v are the same ones as in (0.2) and (0.3), and $\psi = \operatorname{Im} u$.

The following fact is well known.

Proposition 1.1. *Under the parametrization of (1.1), we put $M = P^*P$. Then the Ernst equations (0.2) and (0.3) are equivalent to the following equation:*

$$d(\rho * dMM^{-1}) = 0. \tag{1.2}$$

Moreover the function τ is a solution of (0.4) and (0.5) if and only if it is a solution of the following equations :

$$\tau^{-1}\partial_z\tau = \frac{\rho}{4}\mathrm{tr}(\partial_z MM^{-1}\partial_\rho MM^{-1}) \tag{1.3}$$

$$\tau^{-1}\partial_\rho\tau = \frac{\rho}{8}\mathrm{tr}((\partial_\rho MM^{-1})^2 - (\partial_z MM^{-1})^2). \tag{1.4}$$

The integrability of τ follows easily from (1.3) and (1.4). Equation (1.2) is also called the Ernst equation. We shall consider the solutions satisfying

$$P|_{(z,\rho)=(0,0)} = 1,$$

which corresponds to the gauge fixing condition (0.6).

It is also known that the equation (1.2) can be rewritten as the integrability condition of a 1-form with values in $\mathfrak{g}$ each of whose component is an element of $\mathbb{C}(z,\rho)\otimes_{\mathbb{C}}\mathbb{C}[[t]]$, where $\mathbb{C}(z,\rho)$ is the quotient field of $R = \mathbb{C}[[z,\rho]]$ and t an indeterminate called "spectral parameter". Namely, let $\mathcal{A}$ and $\mathcal{I}$ be 1-forms defined by

$$\mathcal{A} = \frac{1}{2}(dPP^{-1} - (dPP)^*) \quad \mathcal{I} = \frac{1}{2}(dPP^{-1} + (dPP)^*)$$

for any $P \in A_R N_R$, and put

$$\Omega_P = \mathcal{A} + \left(\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}*\right)\mathcal{I},$$

where $*$ is the Hodge operator given by $*dz = d\rho$, $*d\rho = -dz$. We extend the canonical exterior derivative d on $\mathbb{C}(z,\rho)$ to that on $\mathbb{C}(z,\rho)\otimes_{\mathbb{C}}\mathbb{C}[[t]]$ by defining

$$dt = \frac{t}{(1+t^2)\rho}\left((1-t^2)d\rho + 2tdz\right). \tag{1.5}$$

Note then that $d^2 t = 0$. Now we have

Proposition 1.2. *Ω_P satisfies the integrability condition, i.e.,*

$$d\Omega_P - \Omega_P \wedge \Omega_P = 0 \tag{1.6}$$

if and only if P is a solution of (1.2).

It follows from Proposition 1.2 that if P is a solution of the Ernst equation, then there exists a *potential* $p = \sum_{n\geqslant 0} p_n t^n$ such that each entry of p_n is an element of $\mathbb{C}(z,\rho)$ and

$$dp = \Omega_P \cdot p \quad \text{and} \quad p_0 = P. \tag{1.7}$$

2. Hauser Group

We introduce formal loop algebras and formal loop groups, following [T].

Put $F_0 = R = \mathbb{C}[[z,\rho]]$ and $F_n = \rho^{|n|}R$ for a nonzero integer n. We introduce a topology in R by declaring that $\{F_n\}_{n\geqslant 0}$ forms a fundamental neighborhoods system of 0. Note that $F_m F_n \subset F_{m+n}$ for $m,n \geqslant 0$.

Then we define a formal loop algebra $\mathcal{F}\mathfrak{gl}$ by

$$\mathcal{F}\mathfrak{gl} = \left\{ X = \sum_{n\in\mathbb{Z}} X_n t^n \,;\, X_n \in \mathfrak{gl}(N+2, F_n) \right\}. \tag{2.1}$$

Let * be an anti-involution of $\mathcal{F}\mathfrak{gl}$ defined by

$$X^* = \sum_{n\in\mathbb{Z}} X_n^* (-1/t)^n$$

for $X = \sum_{n\in\mathbb{Z}} X_n t^n$. This is well-defined by the definition of our filtration $\{F_n\}_{n\in\mathbb{Z}}$.

We define a formal loop group $\mathcal{F}\mathcal{G}_0$, following [T], by

$$\mathcal{F}\mathcal{G}_0 = \left\{ g = \sum_{n\in\mathbb{Z}} g_n t^n \in \mathcal{F}\mathfrak{gl};\, g^* J g = J,\ \det g = 1,\ g_0|_{(z,\rho)=(0,0)} = 1 \right\} \tag{2.2}$$

and its subgroups by

$$\mathcal{F}\mathcal{K} = \left\{ k = \sum_{n\in\mathbb{Z}} k_n t^n \in \mathcal{F}\mathcal{G}_0 \,;\, \theta^{(\infty)} k = k \right\} \tag{2.3}$$

$$\mathcal{F}\mathcal{P} = \left\{ p = \sum_{n\in\mathbb{Z}} p_n t^n \in \mathcal{F}\mathcal{G}_0 \,;\, p_0 \in A_R N_R,\ p_n = 0 \text{ if } n < 0 \right\}. \tag{2.4}$$

Since $\mathcal{F}\mathcal{G}_0$ is canonically embedded in $\mathcal{F}\mathfrak{gl}$, we can define an involution $\theta^{(\infty)}$ of $\mathcal{F}\mathcal{G}_0$ by

$$\theta^{(\infty)}(g) = (g^*)^{-1} \quad \text{for } g \in \mathcal{F}\mathcal{G}_0,$$

which we call Cartan involution of $\mathcal{F}GL$.

Then, using the Birkhoff decomposition ((3.17), [T]), we can decompose uniquely an element $g \in \mathcal{F}\mathcal{G}$ as

$$g = kp \quad (k \in \mathcal{F}\mathcal{K},\ p \in \mathcal{F}\mathcal{P}). \tag{2.5}$$

Let s be another indeterminate. Define an infinite dimensional group $\mathcal{G}^{(\infty)}$, which we call Hauser group, by

$$\mathcal{G}^{(\infty)} = \left\{ g = \sum_{n\geqslant 0} g_n s^n \in GL(N+2, \mathbb{C}[[s]])\,;\, g^* J g = J, \det g = 1, g_0 = 1 \right\},$$

where $\mathbb{C}[[s]]$ is a ring of formal power series in s over $\mathbb{C}$ and $g^* = \sum g_n^* s^n$.

Let j be a homomorphism of $GL(N+2, \mathbb{C}[[s]])$ into $\mathcal{F}GL$ given by

$$j : g = \sum_{n\geqslant 0} g_n s^n \longmapsto j(g) = \sum_{n\geqslant 0} g_n \left(\rho(\frac{1}{t} - t) + 2z\right)^n .$$

Then it is easy to see that j is injective and that the image of $\mathcal{G}^{(\infty)}$ by j is in $\mathcal{F}\mathcal{G}_0$. We denote by $\mathcal{F}\mathcal{H}$ the image of $\mathcal{G}^{(\infty)}$ by j. The following equations characterize the elements of $\mathcal{F}\mathcal{H}$ in $\mathcal{F}\mathcal{G}$.

Lemma 2.1. *An element $g \in \mathcal{F}\mathcal{G}$ belongs to $\mathcal{F}\mathcal{H}$ if and only if g satisfies the following equations :*

$$\partial_t g = -\rho\left(\partial_z + \frac{1}{t}\partial_\rho\right) g \tag{2.6}$$

$$\partial_t g = -\frac{\rho}{2}\left(1 + \frac{1}{t^2}\right)\partial_z g. \tag{2.7}$$

This characterization will play an important role in the proof of our main theorem.

Definition. Let $\mathcal{F}\mathcal{P}$ be as in (2.4). We define $\mathcal{S}\mathcal{P}$ to be a subset of $\mathcal{F}\mathcal{P}$ consisting of elements $p = \sum_{n\geqslant 0} p_n t^n$ which satisfy the following conditions :

$$dp = \Omega_{p_0} \cdot p \quad \text{and} \quad p_0|_{(z,\rho)=(0,0)} = 1. \tag{2.8}$$

We call $\mathcal{S}\mathcal{P}$ *the space of potentials.*

It follows from (2.8) that p_0 is a solution of the Ernst equation (1.2) for $p = \sum_{n\geqslant 0} p_n t^n \in \mathcal{S}\mathcal{P}$.

Theorem 2.2. *Let $p \in \mathcal{F}\mathcal{P}$. Then $p \in \mathcal{S}\mathcal{P}$ if and only if $p^* p \in \mathcal{F}\mathcal{H}$.*

Let $p \in \mathcal{S}\mathcal{P}$ and $g \in \mathcal{G}^{(\infty)}$. By (2.5) there exist $k \in \mathcal{F}\mathcal{K}$ and $p_g \in \mathcal{F}\mathcal{P}$ such that

$$p \cdot j(g) = k^{-1} \cdot p_g. \tag{2.9}$$

Then, it follows immediately from Theorem 2.2 that p_g is in $\mathcal{S}\mathcal{P}$. Thus we can define an action of the Hauser group $\mathcal{G}^{(\infty)}$ on $\mathcal{S}\mathcal{P}$ to the right by

$$\mathcal{S}\mathcal{P} \times \mathcal{G}^{(\infty)} \longrightarrow \mathcal{S}\mathcal{P} \quad (p, g) \longmapsto p_g, \tag{2.10}$$

where p_g is given by (2.9).

From the fact that an element $g = \sum_{n\geqslant 0} g_n s^n \in \mathcal{G}^{(\infty)}$ such that $g^* = g$ and such that g_0 is positive definite decomposes as $g = h^* h$ for some $h \in \mathcal{G}^{(\infty)}$, we have

Corollary 2.3. *The action of $\mathcal{G}^{(\infty)}$ on $\mathcal{SP}$ given by (2.10) is transitive.*

Remark. As we mentioned in [S], our group $\mathcal{G}^{(\infty)}$ is too small to obtain all solutions of the Ernst equation (1.2) through the action (2.10).

3. 2-Cocycle on $\mathcal{FG}_0$

The formal loop algebra $\mathcal{F}\mathfrak{gl}$ becomes a Lie algebra with Lie bracket $[X, Y] = XY - YX$. The map

$$\exp : \mathcal{F}\mathfrak{gl} \longrightarrow \mathcal{F}GL$$

given by

$$\exp X = e^X = \sum_{n \geqslant 0} \frac{X^n}{n!} \tag{3.1}$$

is called the *formal exponential map.* Note that for any $g \in \mathcal{FG}_0$ we can find a unique element X in $\mathcal{F}\mathfrak{gl}$ such that $g = e^X$, since the *logarithm* given by

$$\log(1 + A) = \sum_{n \geqslant 1} \frac{(-1)^{n-1}}{n} A^n \tag{3.2}$$

is well-defined and satisfies

$$e^{\log(1+A)} = 1 + A \tag{3.3}$$

for $A = \sum_{n \in \mathbf{Z}} a_n t^n \in \mathcal{F}\mathfrak{gl}$ with $a_0 \in \mathfrak{gl}(N+2, \mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal of R.

For X, Y in $\mathcal{F}\mathfrak{gl}$, let $c_n(X, Y)$ $(n = 1, 2, \cdots)$ be the elements in $\mathcal{F}\mathfrak{gl}$ which are determined by

$$\exp vX \exp vY = \exp \sum_{n \geqslant 0} c_n(X, Y) v^n,$$

where v is an indeterminate. Furthermore c_n's are uniquely determined by the following recursion formulas (see [V]) :

$$\begin{aligned} c_1(X, Y) &= X + Y \\ (n+1)c_{n+1}(X, Y) &= \frac{1}{2}[X - Y, c_n(X, Y)] \\ &+ \sum_{p \geqslant 1, 2p \leqslant n} K_{2p} \sum_{\substack{k_1, \cdots, k_{2p} > 0 \\ k_1 + \cdots + k_{2p} = n}} [c_{k_1}(X, Y), [\cdots, [c_{k_{2p}}(X, Y), X + Y] \cdots] \quad (n \geqslant 1), \end{aligned}$$

where K_{2p}'s are determined by

$$\frac{x}{1 - e^{-x}} - \frac{1}{2}x = 1 + \sum_{p \geqslant 1} K_{2p} x^{2p}.$$

We set $C(X, Y) = \sum_{n \geqslant 1} c_n(X, Y)$. Then $C(X, Y)$ is a well-defined element of $\mathcal{F}\mathfrak{gl}$ for X, Y such that $X_0, Y_0 \in \mathfrak{gl}(N+2, \mathfrak{m})$.

Lemma 3.1. *For $n \geqslant 2$, there exists a $\mathcal{F}\mathfrak{gl}$-valued function $L_n(\cdot,\cdot)$ which satisfies*

$$c_n(X,Y) = [X, L_n(X,Y)] + [Y, L_n(-Y,-X)]. \tag{3.4}$$

for $X, Y \in \mathcal{F}\mathfrak{gl}$.

Note that L_n's are not uniquely determined, however, we fix L_n's so that there holds

$$L(X, vY) = \left(\frac{e^{-\mathrm{ad}X} - 1 + \mathrm{ad}X}{\mathrm{ad}X(1 - e^{-\mathrm{ad}X})} - \frac{1}{4} \right) vY + O(v^2), \tag{3.5}$$

where we put $L(X,Y) = \sum_{n\geqslant 2} L_n(X,Y)$ for $X, Y \in \mathcal{F}\mathfrak{gl}$ such that $X_0, Y_0 \in \mathfrak{gl}(N+2, \mathfrak{m})$. Thus, we obtain

$$C(X,Y) = X + Y + [X, L(X,Y)] + [Y, L(-Y,-X)].$$

For a series $f = \sum_{n\in\mathbb{Z}} f_n t^n \in R[[t, t^{-1}]]$, we write

$$\mathrm{Res}_t\, f = f_{-1} \in R.$$

Let $R_0 = \mathbf{R}[[z, \rho]] \subset R$, the formal power series in z and ρ over $\mathbf{R}$. We define a R_0-valued 2-cocycle ω on $\mathcal{F}\mathfrak{gl}$ by

$$\omega(X,Y) = \mathrm{Res}_t\, \mathrm{Re}\,\mathrm{tr} X \partial_t Y$$

for $X, Y \in \mathcal{F}\mathfrak{gl}$. Note that

$$\omega(X^*, Y^*) = -\omega(X,Y) \tag{3.6}$$

for $X, Y \in \mathcal{F}\mathfrak{gl}$.

Now we introduce a group 2-cocycle on $\mathcal{F}\mathcal{G}_0$, following [BM]. Note that, from (3.3), any element $g \in \mathcal{F}\mathcal{G}_0$ can be uniquely written as $g = e^X$ for $X \in \mathcal{F}\mathfrak{gl}$ with $X_0 \in \mathfrak{gl}(N+2, \mathfrak{m})$.

Definition. Let Ξ be a R_0-valued function on $\mathcal{F}\mathcal{G}_0 \times \mathcal{F}\mathcal{G}_0$ defined by

$$\Xi(e^X, e^Y) = \omega(X, L(X,Y)) + \omega(Y, L(-Y,-X)).$$

Then Ξ defines a 2-*cocycle* on $\mathcal{F}\mathcal{G}_0$, i.e. satisfies the cocycle condition :

$$\Xi(e^X, e^Y) + \Xi(e^X e^Y, e^Z) = \Xi(e^Y, e^Z) + \Xi(e^X, e^Y e^Z) \tag{3.7}$$

for $X, Y, Z \in \mathcal{F}\mathfrak{gl}$.

4. Central Extension

For any $p \in \mathcal{SP}$, we can find an element $g \in \mathcal{FH}$ which sends the identity element $1 \in \mathcal{SP}$ to p by Corollary 2.2. Then we have $p = kg$ for some $k \in \mathcal{FK}$.

Proposition 4.1. *For $p = \sum_{n\geqslant 0} p_n t^n \in \mathcal{SP}$, let $g \in \mathcal{FH}$ and $k \in \mathcal{FK}$ be such that $p = kg$. Let τ be a solution of (1.3) and (1.4) corresponding to $P = p_0$. Then we have the following relations:*

$$\tau^{-1}\partial_z\tau = \partial_z\Xi(kg, g^{-1}) \tag{4.1}$$
$$\tau^{-1}\partial_\rho\tau = \partial_\rho\Xi(kg, g^{-1}). \tag{4.2}$$

Now we define a central extension of $\mathcal{FG}_0$ in terms of the cocycle Ξ.

Definition. Let $(\mathcal{FG}_0)^{\sim}$ be the set given by

$$(\mathcal{FG}_0)^{\sim} = \{(g, e^{\mu});\ g \in \mathcal{FG}_0, \mu \in R_0\}.$$

Define a product of any two elements of $(\mathcal{FG}_0)^{\sim}$ by

$$(g_1, e^{\mu_1}) \cdot (g_2, e^{\mu_2}) = (g_1 g_2, e^{\mu_1+\mu_2+\Xi(g_1,g_2)}) \tag{4.3}$$

for $(g_1, e^{\mu_1}), (g_2, e^{\mu_2}) \in (\mathcal{FG}_0)^{\sim}$. Since Ξ satisfies the cocycle condition (3.7), $(\mathcal{FG}_0)^{\sim}$ forms a group with group multiplication given by (4.3). Namely, $(\mathcal{FG}_0)^{\sim}$ is a *central extension* of $\mathcal{FG}_0$.

Let $\tilde{\theta}^{(\infty)}$ be an involution of $(\mathcal{FG}_0)^{\sim}$ given by

$$\tilde{\theta}^{(\infty)}(g, e^{\mu}) = (\theta^{(\infty)}(g), e^{-\mu}).$$

If we denote by $(\mathcal{FK})^{\sim}$ the subgroup of $(\mathcal{FG}_0)^{\sim}$ consisting of elements whish are fixed by $\tilde{\theta}^{(\infty)}$, then we have

$$(\mathcal{FK})^{\sim} = \{(k, 1) \in (\mathcal{FG}_0)^{\sim};\ k \in \mathcal{FK}\}.$$

Let $(\mathcal{FP})^{\sim}$ be a subgroup of $(\mathcal{FG}_0)^{\sim}$ given by

$$(\mathcal{FP})^{\sim} = \{(p, e^{\mu}) \in (\mathcal{FG}_0)^{\sim};\ p \in \mathcal{FP}, \mu \in R_0\}.$$

It follows immediately from the decomposition (2.5) of $\mathcal{FG}$ that $(\mathcal{FG}_0)^{\sim}$ has a unique decomposition :

$$(\mathcal{FG}_0)^{\sim} = (\mathcal{FK})^{\sim} \cdot (\mathcal{FP})^{\sim}. \tag{4.4}$$

Furthermore, we put

$$(\mathcal{FH})^{\sim} = \{(g, e^{\gamma}) \in (\mathcal{FG}_0)^{\sim};\ g \in \mathcal{FH},\ \gamma \in \mathbb{R}\}.$$

It follows from Lemma 3.2, [HS2] that $\mathcal{FH}$ can be regarded as a subgroup of $(\mathcal{FH})^{\sim}$ by

$$\mathcal{FH} \longrightarrow (\mathcal{FH})^{\sim}, \quad g \longmapsto (g, 1).$$

Let $(\mathcal{SP})^{\sim}$ be the subset of $(\mathcal{FP})^{\sim}$ given by

$$(\mathcal{SP})^{\sim} = \Big\{ (p, e^{\mu}) \in (\mathcal{FP})^{\sim}; \, p = \sum_{n \geqslant 0} p_n t^n \in \mathcal{SP},$$

$$\tau = e^{-\mu} \text{ satisfies (1.3) and (1.4) with } P = p_0 \Big\}. \tag{4.5}$$

We call $(\mathcal{SP})^{\sim}$ *the space of potentials with conformal factor.*

Proposition 4.2. *For $p \in \mathcal{SP}$, let $k \in \mathcal{FK}$ and $g \in \mathcal{FH}$ be as above, i.e. $p = kg$. Then we have*

$$\Xi(p^*, p) = 2\,\Xi(kg, g^{-1}). \tag{4.6}$$

Therefore, any element of $(\mathcal{SP})^{\sim}$ can be written as $(p, e^{-\frac{1}{2}\Xi(p^*,p)+\gamma})$ for $p \in \mathcal{SP}, \gamma \in \mathbb{R}$.

Define an action of $(\mathcal{FH})^{\sim}$ on the space of potentials with conformal factor $(\mathcal{SP})^{\sim}$ to the right through the decomposition (4.4) :

$$(\mathcal{SP})^{\sim} \times (\mathcal{FH})^{\sim} \longrightarrow (\mathcal{SP})^{\sim}, \quad ((p, e^{\mu}), (g, e^{\gamma})) \longmapsto (p_g, e^{\alpha}). \tag{4.7}$$

Namely, we can find a unique element $(k, 1) \in (\mathcal{FK})^{\sim}$ and $(p_g, e^{\alpha}) \in (\mathcal{FP})^{\sim}$ such that

$$(p, e^{\mu})(g, e^{\gamma}) = (k, 1)^{-1}(p_g, e^{\alpha}),$$

where k and p_g are the elements given in (2.9). Since we have

$$\tilde{\theta}^{(\infty)}((p, e^{\mu})(g, e^{\gamma}))^{-1} \cdot (p, e^{\mu})(g, e^{\gamma}) = (g^* p^* p g, e^{2(\mu+\gamma)+\Xi(p^*,p)})$$

and

$$\tilde{\theta}^{(\infty)}(p_g, e^{\alpha})^{-1} \cdot (p_g, e^{\alpha}) = (p_g^* p_g, e^{2\alpha+\Xi(p_g^*,p_g)}),$$

we obtain

$$\begin{aligned} \alpha =& \mu + \gamma + \frac{1}{2}(\Xi(p^*, p) - \Xi(p_g^*, p_g)) \\ =& \gamma' - \frac{1}{2}\Xi(p_g^*, p_g) \end{aligned}$$

for some $\gamma' \in \mathbb{R}$, where we used Proposition 4.4. Thus (p_g, e^{α}) belongs to $(\mathcal{SP})^{\sim}$, i.e. the action (4.7) of $(\mathcal{FH})^{\sim}$ is well-defined.

Now we state our main theorem :

Theorem 4.3. *The group $(\mathcal{FH})^{\sim}$ acts transitively on the space of potentials with conformal factor $(\mathcal{SP})^{\sim}$ by (4.7).*

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